Transcendence degrees of fields generated by exponentials of products

Heinrich Massold

June 1, 2025

Abstract

Let $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{R}^m, \kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{R}^n$ be two tuples of real numbers each linearly independent over $\mathbb{Q}$, and T the transcendence degree of the field generated by $\{\exp(\theta_i \kappa_j) | i = 1, \ldots, m,\ j = 1, \ldots, n\}$ over $\mathbb{Q}$. The estimate $T \geq \frac{mn}{m+n} - 1$ has been conjectured for some time but could only be proved under additional hypotheses for θ and κ. This paper proves a weaker estimate for T while also reducing the strong estimate to a prominent conjecture on intersections of subvarieties of split tori with subgroups.

Contents

1 Introduction

Let l be a natural number, G the algebraic group $G = (\mathbb{G}_m)^l_{\mathbb{Z}}$ and $X \subset G$ an irreducible subvariety defined over an algebraic extension of $\mathbb{Q}$ that is contained in no proper algebraic subgroup of G. With $t = \dim X$, an irreducible subvariety $Y \subset X$ of dimension s is called special if Y is contained in an algebraic subgroup $H \subset G$ of codimension $t + 1 - s$.

1.1 Conjecture *In the situation above, there is a proper Zariski closed subset $\bar{X} \subset X$ that contains all special subvarieties.*

This conjecture is proved for $t = 1$ ([Mau], Théorème 1.2).

For a finite set $\Theta = \{\Theta_1, \ldots, \Theta_n\} \subset \mathbb{C}$, denote the transcendence degree of $\mathbb{Q}(\Theta)$ over $\mathbb{Q}$ by $T(\Theta)$.
A tuple of complex numbers $(\theta_1, \ldots, \theta_m) \in \mathbb{C}^m$ will be called regular if the point $(\exp(\theta_1), \ldots, \exp(\theta_m)) \in \mathbb{G}_m^m(\mathbb{C})$ is not contained in any proper algebraic subgroup, which is equivalent to $\theta_1, \ldots, \theta_m, \pi i$ being linearly independent over $\mathbb{Q}$.
If $\theta_1, \ldots, \theta_m \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$, then $(\theta_1, \ldots, \theta_m)$ is regular. If $\theta_1, \ldots, \theta_m \in \mathbb{C}$ are linearly independent over $\mathbb{Q}$ but $(\theta_1, \ldots, \theta_m)$ is not regular, then there is a subset $\{\theta_{i_1}, \ldots, \theta_{i_{m-1}}\} \subset \{\theta_1, \ldots, \theta_m \in \mathbb{C}\}$ of $m-1$ numbers such that $\theta_{i_1}, \ldots, \theta_{i_{m-1}}, \pi i$ are linearly independent over $\mathbb{Q}$ and

$$T(\exp(\theta_{i_1}), \ldots, \exp(\theta_{i_{m-1}})) = T(\exp(\theta_1), \ldots, \exp(\theta_m)).$$

1.2 Definition *Let $n, m, \nu, \mu \in \mathbb{N}, \nu \leq n, \mu \leq m, \eta > 1$, and $\theta = (\theta_1, \ldots, \theta_m) \in \mathbb{C}^m$ and $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{C}^n$ be regular.*

1. *The m-tuple $\theta = \{\theta_1, \ldots, \theta_m\}$ is called (μ, η)-generic, if there is a nonsingular $m \times m$-matrix A with entries in $\mathbb{Q}$ such that for every sufficiently big $D \in \mathbb{N}$, there are $1 \leq i_1 < \cdots < i_\mu \leq m$ such that for every nonzero $l = (l_1, \ldots, l_\mu) \in \mathbb{Z}^\mu$ with $|l| \leq D$, the inequality*

   $$\log|\exp((l_1 A\theta)_{i_1} + \cdots + l_\mu (A\theta)_{i_\mu}) - 1| \gg -D^\eta$$

   holds, where the implied constant depends only on θ and A.

   The m-tuple θ is called (μ, η)-special if it is not (μ, η)-generic, i.e. if for every $c > 0$ and every nonsingular $A \in M_{m\times m}(\mathbb{Q})$, there are infinitely many $D \in \mathbb{N}$ such that for every $\{i_1, \ldots, i_\mu\} \subset \{1, \ldots, m\}$ there is a nonzero $l = (l_1, \ldots, l_\mu) \in \mathbb{Z}^\mu$ with $|l| \leq D$ and

   $$|\exp(l_1(A\theta)_{i_1} + \cdots l_\mu(A\theta)_{i_\mu}) - 1| \leq -cD^\eta.$$

   Denote by gen(θ, η) the biggest number $\nu \in \mathbb{N}$ such that θ is (ν, η)-generic.

2. *The bituple $((\theta_1, \ldots, \theta_m), (\kappa_1, \ldots, \kappa_n))$ is called (μ, ν, η)-generic if there are regular matrices $A \in M_{m\times m}(\mathbb{Q}), B \in M_{n\times n}(\mathbb{Q})$ and a $D_0 \in \mathbb{N}$ such that for all $L, R \geq D_0$, there are subsets $\{i_1, \ldots, i_\mu\} \subset \{1, \ldots, m\}$, $\{j_1, \ldots, j_\nu\} \subset \{1, \ldots, n\}$ such that for all nonzero $l = (l_1, \ldots, l_\mu) \in \mathbb{Z}^\mu, r = (r_1, \ldots, r_\nu) \in \mathbb{Z}^\nu$ with $|l| \leq L$ and $|r| \leq r$, the inequality*

   $$\log|\exp((l_1(A\theta)_{i_1} + \cdots l_\mu(A\theta)_{i_\mu})(r_1(B\kappa)_{j_1} + \cdots r_\nu(B\kappa)_{j_\nu})) - 1| \gg -L^\eta - R^\eta$$

holds, where the implied constant depends only on θ, κ, A, and B.

The bituple $((\theta_1, \dots, \theta_m), (\kappa_1, \dots, \kappa_n))$ is called (μ, ν, η)-special if it is not (μ, ν, η)-generic, i.e. if for every $c > 0$ and every $A \in M_{n\times n}(\mathbb{Q}), B \in M_{n\times n}(\mathbb{Q})$ with nonzero determinant, there are arbitrarily big $L, R \in \mathbb{N}$ such that for every $\{i_1, \dots, i_\mu\} \subset \{1, \dots, n\}$ there are nonzero $l = (l_1, \dots, l_\mu) \in \mathbb{Z}^\mu, r = (r_1, \dots, r_\nu) \in \mathbb{Z}^\mu$ with $|l| \le L, |r| \le R$, and

$$\log|\exp((l_1(A\theta)_{i_1} + \cdots l_\mu(A\theta)_{i_\mu})(r_1(B\kappa)_{j_1} + \cdots r_\nu(B\kappa)_{j_\nu})) - 1| \le -c(L^\eta + R^\eta)$$

1.3 Conjecture *For $m, n \in \mathbb{N}$, let $\theta = (\theta_1, \dots, \theta_n) \in \mathbb{C}^m$ and $\kappa = (\kappa_1, \dots, \kappa_n) \in\in \mathbb{C}^n$ be regular $\mathbb{Q}$.*

1. *If gen$(\theta, t) \le m - t$ for some $t \in \mathbb{N}$, then $T(\exp(\theta_1), \dots, \exp(\theta_m)) \ge t$, and if gen$(\kappa, t) \le n - t$ for some $t \in \mathbb{N}$, then $T(\exp(\kappa_1), \dots, \exp(\kappa_n)) \ge t$.*

2. *If either gen$(\theta, t) < m$, and $n \ge t$, or gen$(\kappa, t) < n$, and $m \ge t$ for some $t \in \mathbb{N}$, then*

$$T(\{\exp(\theta_i\kappa_j)|i = 1, \dots, m,\ j = 1, \dots, n\}) \ge t.$$

3. *If $\theta_1, \dots, \theta_n, \kappa_1, \dots, \kappa_m$ are all real, then*

$$T(\{\exp(\theta_i\kappa_j)|i = 1, \dots, m,\ j = 1, \dots, n\}) \ge \frac{mn}{m+n} - 1,$$

for every $\epsilon > 0$.

1.4 Theorem I *Conjecture 1.1 implies conjecture 1.3*

1.5 Theorem II *For $m, n \in \mathbb{N}$, let $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{C}1m$ and $\kappa_1, \dots, \kappa_n \in \mathbb{C}^n$ be regular.*

1. *If gen$(\theta, t) \le \frac{m-t}{\max(t-1,1)}$ for some $t \in \mathbb{N}$, then $T(\exp(\theta_1), \dots, \exp(\theta_m)) \ge t$. Likewise, if gen$(\kappa, t) \le \frac{n-t}{\max(t-1,t)}$ for some $t \in \mathbb{N}$, then $T(\exp(\kappa_1), \dots, \exp(\kappa_n)) \ge t$.*

2. *For $\theta_1, \dots, \theta_n, \kappa_1, \dots, \kappa_m$ all real, $t, m, n \in \mathbb{N}$, if there exist $\mu, \nu \in \mathbb{N}$ with $\mu \le m$, and $\nu \le n$, such that*

$$\frac{\mu\nu}{\mu+\nu} > t, \quad \mu \le \frac{m-t}{\max(t-1,1)}, \quad \text{and} \quad \nu \le \frac{n-t}{\max(t-1,1)},$$

then

$$T(\{\exp(\theta_i\kappa_j)|i = 1, \dots, m,\ j = 1, \dots, n\}) \ge t.$$

The above three inequalities are fulfilled e. g. for $t \ge 2$ arbitrary, $\mu = 2t+1, \nu = 2t, m \ge 2t^2 - 1$, and $n \ge 2t^2 - t$.

1.6 Corollary

If $\theta_1, \ldots, \theta_m \in \mathbb{R}$ as well as $\kappa_1, \ldots, \kappa_n \in \mathbb{R}$ are linearly independent over $\mathbb{Q}$, then

$$T(\{\exp(\theta_i \kappa_j) | i = 1, \ldots, m, \ j = 1, \ldots, n\}) \geq \left[\sqrt{\frac{min(m,n)+1}{2}}\right].$$

If $\zeta \in \mathbb{R}$ is a transcendental number, and $m, n \in \mathbb{Z}$, then

$$T(\{\exp(\zeta^m), \exp(\zeta^{m+1}), \ldots, \exp(\zeta^{n-m})\}) \geq \left[\sqrt{\frac{n}{4}+\frac{1}{2}}\right].$$

Proof 1. Assume $\min(m,n) = n$. In Theorem II.2 take $t = \left[\sqrt{\frac{n+1}{2}}\right]$, $\mu = \nu = 2t^2 - 1$. An easy calculation shows that $m \geq n \geq (\nu - 1)(t-1) - 1$, and $\frac{\mu\nu}{\mu+\nu} > t$, and the estimate $T(\{\exp(\theta_i \kappa_j) | i = 1, \ldots, m, \ j = 1, \ldots, n\}) \geq t = \left[\sqrt{\frac{n+1}{2}}\right]$ follows.

2. In part one, take $\theta = (\zeta^{\left[\frac{m}{2}\right]}, \ldots, \zeta^{(n-m)-\left[\frac{n-m}{2}\right]})$, and $\kappa = (\zeta^{m-\left[\frac{m}{2}\right]}, \ldots, \zeta^{\left[\frac{n-m}{2}\right]})$.

Throughout the whole paper, the norm of a polynomial with coefficients in $\mathbb{Z}$ will always be the maximum norm, e. g. for $f = \sum_{i=0}^n a_i x^i$, we have $|f| = \max_{i=0,\ldots,n} |a_i|$. To prove estimates for transcendence degrees, two special cases of the Philippon criterion will be needed.

1.7 Proposition

Let $\Theta = (\Theta_1, \ldots, \Theta_n) \in \mathbb{C}^n$, and $c_1, c_2, \eta > 0, c_3 \in \mathbb{R}$, and denote by $|z_1, z_2|$ the distance of the point z_1, z_2 in $\mathbb{C}^n$. There is a constant $C \gg 0$, only depending on n, Θ, c_1, c_2 and η such that

1. *if for every sufficiently big natural number D, there are polynomials $f_1, \ldots, f_m$ in n variables, such that*

   $$\deg f_i \leq c_1 D, \quad \log|f_i| \leq c_2 D, \quad \log|f_i(\Theta_1, \ldots, \Theta_n)| \leq -CD^\eta, \quad i = 1, \ldots, m,$$

   and for every common zero $z = (z_1, \ldots, z_n)$ of $f_1, \ldots, f_m$, we have $\log|z, \Theta| \geq -3CD^\eta$. Then, $T(\Theta) > \eta - 1$.

2. *if for infinitely many natural numbers D, there are polynomials $f_1, \ldots, f_m$ in n variables, such that*

   $$\deg f_i \leq c_1 D, \quad \log|f_i| \leq c_2 D, \quad \log|f_i(\Theta_1, \ldots, \Theta_n)| \leq -CD^\eta, \quad i = 1, \ldots, m,$$

   and for every common zero $z = (z_1, \ldots, z_n)$ of $f_1, \ldots, f_m$, we have $\log|z, \Theta| \geq c_3$. Then, $T(\Theta) > \eta - 1$.

Proof This follows from [Ph], Théorème 2.11.

2 Generic points

2.1 Lemma

1. *For all θ, η, we have $1 \leq gen(\theta, \eta) \leq n$.*
2. *For a nonsingular $A \in M_{m\times m}(\mathbb{Q})$, the tuple $(\theta_1, \dots, \theta_m)$ is (μ, η)-regular if and only $(A\theta_1, \dots, A\theta_m)$ is.*
3. *For a nonzero $a \in \mathbb{R}$ the tuple $(\theta_1, \dots, \theta_m) \in \mathbb{R}^m$ is (μ, η)-regular if and only $(a\theta_1, \dots, a\theta_m)$ is.*
4. *For nonsingular $A \in M_{m\times m}(\mathbb{Q}), B \in M_{n\times n}(\mathbb{Q})$, the bituple (θ, κ) is (μ, ν, η)-regular if and only $(A\theta, B\kappa)$ is.*
5. *For $A \in M_{s\times m}(\mathbb{Q})$ a matrix of rank s,*
$$gen(A\theta, t-1) \leq gen(\theta, t-1) \leq gen(\theta, t).$$
6. *For $\theta_1, \dots, \theta_m, \kappa_1, \dots, \kappa_n \in \mathbb{R}$, if $\theta = (\theta_1, \dots, \theta_m)$ is (μ, η)-regular and $\kappa = (\kappa_1, \dots, \kappa_n)$ is (ν, η)-regular, then the bituple (θ, κ) is (μ, ν, η)-regular.*

PROOF 1. Obvious.

2,.3.,4. These claims hold since the relation $\gg$ remains true if one side is changed by a fixed multiplicative constant.

5. Let $\mu = \text{gen}(A\theta, t-1)$. Then for every sufficiently big D and every $I = (i_1, \dots, i_\mu)$ with $1 \leq i_1 < \cdots < i_\mu \leq s$ and every $l = (l_1, \dots, l_\mu)$ with $|l| \leq D$, the inequality

$$\log|l_1(A\theta)_{i_1} + \cdots + l_\mu(A\theta)_{i_\mu} - 1| \gg -D^t.$$

holds. If A is extended to an $(m \times m)$ matrix A' of rank n, then

$$\log|l_1(A'\theta)_{i_1} + \cdots + l_\mu(A'\theta)_{i_\mu} - 1| \gg -D^t,$$

for all $I = (i_1, \dots, i_\mu)$ with $1 \leq i_1 < \cdots < i_\mu \leq s$ and all l, since the left hand side is unchanged because of $i_\mu \leq s$. Hence, $\text{gen}(A\theta, t-1) = \mu \leq \text{gen}(A'\theta, t-1)$. Further, $\text{gen}(A'\theta, t-1) = \text{gen}(\theta, t-1)$ by part 2 of the Lemma, and the inequality $\text{gen}(\theta, t-1) \leq \text{gen}(\theta, t)$ trivially holds.

6. Since $\exp : \mathbb{R} \to \mathbb{R}^+$ is a bijection with $\exp(0) = \exp'(0) = 1$, the relations

$$\log|l_1 \exp((A\theta)_{i_1} + \cdots + l_\mu(A\theta)_{i_\mu}) - 1| \gg -L^\eta,$$

$$\log|l_1 \exp((B\kappa)_{j_1} + \cdots + l_\mu(B\kappa)_{j_\nu}) - 1| \gg -R^\eta$$

are equivalent to

$$\log|l_1(A\theta)_{i_1}+\cdots+l_\mu(A\theta)_{i_\mu}|\gg -L^\eta,$$
$$\log|l_1(B\kappa)_{j_1}+\cdots+l_\mu(B\kappa)_{j_\nu}|\gg -R^\eta,$$

which in turn implies

$$\log|((l_1(A\theta)_{i_1}+\cdots l_\mu(A\theta)_{i_\mu})(r_1(B\kappa)_{j_1}+\cdots r_\nu(B\kappa)_{j_\nu})|\gg -L^\eta-R^\eta,$$

which again is equivalent to

$$\log|\exp((l_1(A\theta)_{i_1}+\cdots l_\mu(A\theta)_{i_\mu})(r_1(B\kappa)_{j_1}+\cdots r_\nu(B\kappa)_{j_\nu}))-1|\gg -L^\eta-R^\eta.$$

2.2 Proposition *If $\theta=(\theta_1,\ldots,\theta_m)$ as well as $\kappa=(\kappa_1,\ldots,\kappa_n)$ are tuples of linearly independent over $\mathbb{Q}$, and for some $\eta>1$, the inequality*

$$\frac{gen((\theta),\eta)\ gen((\kappa),\eta)}{gen((\theta),\eta)+gen((\kappa),\eta)}>\eta$$

holds, then $T(\{\exp(\theta_i\kappa_j)|i=1,\ldots,n,\ j=1,\ldots,m\})\geq\eta-1$.

For $\mathrm{gen}(\theta)=m$, and $\mathrm{gen}(\kappa)=n$, this is proved in [LNM 1752], chapter 14.3. The general case can be proved in exactly the same way; the only thing that has to be modified is that for every value of the approximation parameter D, one works with only μ, respectively ν components of θ and κ, which may be different components for every D. The necessary auxiliary polynom for each D is then a polynomial in only these "active" variables. As this requires notational and some technical adjustments throughout the whole proof, for the convenience of the reader, I will give a full generalized proof in the appendix.

3 Special points

Let $G=\mathbb{G}_m^l$, and $X\subset\mathbb{G}_m$ an irreducible subvariety defined over a finite extension of $\mathbb{Q}$, that is contained in no proper subgroup of G. For M the character module of G, and $N\subset M$ a submodule define the subgroup

$$H_N:=\bigcap_{\chi\in N}\ker\chi\subset G,$$

and for a subvariety $Y\subset X$ define H_Y as the smallest subgroup of G that contains Y.

3.1 Definition *For $s\geq t=\dim X$,*

1. *the variety X is called s-regular, if for every submodule $N \subset M$ of rank s, we have* $\dim((\chi_1,\ldots,\chi_s)(Y)) = t$*, where $\chi_1,\ldots,\chi_s$ is any basis of N, and $(\chi_1,\ldots,\chi_s)$ is the corresponding map $\mathbb{G}_m^n \to \mathbb{G}_m^s$.*

2. *an irreducible subvariety $Y \subset X$ of dimension d is called s-special if* $\dim H_Y < l - s + d$*.*

Conjecture 1.1 says that there is a proper Zariski closed subset $\bar{X} \subset X$ that contains all t-special subvarieties of X.

3.2 Theorem *If $X \subset \mathbb{G}_m^n$ is s-regular for some $s \geq t = \dim X$, then there is a proper Zariski closed subset $\bar{X} \subset X$ that contains all s-special subvarieties of X.*

PROOF This result is a mainly technical generalization of [Ha], Corollary 3 and can be found in [Ma].

3.3 Proposition *Let $\Theta \in G(\mathbb{C})$ be a point that is contained in no proper algebraic subgroup, and X the algebraic closure of $\{\Theta\}$ over $\mathbb{Q}$. Suppose that for $t \in \mathbb{N}$, some $c_1 \geq 1$, an arbitrary $c > 0$, and a proper Zariski closed subset $\bar{X} \subset X$, there is an infinite set $\mathcal{D} \subset \mathbb{N}$ such that for every $D \in \mathcal{D}$ there is a submodule $N_D \subset M$ such that $X \cap H_{N_D} \subset \bar{X}$, and for some basis $\chi_1,\ldots,\chi_r$ of N_D the inequalities*

$$\deg \chi_i \leq c_1 D, \quad \textit{and} \quad \log|\chi_i(\Theta) - 1| \leq -cD^t, \quad i = 1,\ldots,r$$

hold. Then, $T(\Theta) \geq t$.

PROOF As $\bar{X}(\mathbb{C})$ does not contain Θ, the distance $\bar{X}(\mathbb{C})$ to Θ is positive. Denote this distance by c_3, and let $\mathcal{F} = \{f_1,\ldots,f_l\}$ be a set of generators of the ideal of X in G. By shrinking $\mathcal{F}$, if necessary we may assume $\deg f_i \leq D$, and $\log|f_i| \leq D$ for all $i = 1,\ldots,l$, and every $D \in \mathcal{D}$. For $D \in \mathcal{D}$ let $\mathcal{F}_D := \mathcal{F} \cup \{\chi_1 - 1,\ldots,\chi_r - 1\}$. Then, by assumption, $\bar{X}$ contains the set of common zeros of $\mathcal{F}_D$, hence every common zero of $\mathcal{F}_D$ has distance at least c_3 to Θ. Also, for $f \in \mathcal{F}_D$,

$$\deg f \leq c_1 D, \quad \log|f| \leq D, \quad \text{and} \quad \log|f(\Theta)| \leq -cD^t,$$

the second inequality because for f a character, $|f| \leq 1$, the last inequality because $\log|\chi_i(\Theta)| - 1 \leq -cD^t$ by assumption for all $i = 1,\ldots,r$, and $|f(\Theta)| = 0$ for $f \in \mathcal{F}$. By Proposition 1.7.2, $T(\Theta) > t - 1$, from which $T(\Theta) \geq t$ follows, since $T(\Theta)$ is integral.

4 Proof of the main Theorems

4.1 Lemma

1. *Let V be a $\mathbb{Q}$-vector space with basis $\{v_1,\dots,v_n\}$, and ν a natural number less or equal n. For every subset $I \subset \{1,\dots,n\}$ with $|I| = n-\nu$ choose a non-zero vector $w_I = \sum_{i\in I} a_i v_i$. Then, the subpsace W_I generated by these w_I has dimension at least $\nu+1$. More specifically, there are subsets $I_1,\dots I_{\nu+1}$ such that $w_{I_j}, j = 1,\dots,\nu+1$ are linearly independent.*

2. *Let $(\theta_1,\dots,\theta_m) \in \mathbb{R}^m, (\kappa_1,\dots,\kappa_n) \in \mathbb{R}^n$ be two tuples of real numbers, linearly independent over $\mathbb{Q}$, and $\bar{G} \subset G = G_m^{mn}$ the smallest algebraic subgroup that contains $(\exp(\theta_i\kappa_j)_{i\le m, j\le n}$. Further, with $x_{ij}, 1 \le i \le m, 1 \le j \le n$ the coordinate functions of $\mathbb{G}_m^{mn}$, and $l_1,\dots,l_n \in \mathbb{Z}$, not all zero, let $\chi_j, j = 1,\dots,n$ be the characters $\chi_j = \prod_{i=1}^m x_{ij}^{l_i}$, and $H = \bar{G} \cap_{j=1}^n \ker \chi_j$. Then, the codimension of H in $\bar{G}$ equals n.*

Proof 1. The w_{I_j} are construed inductively: Take w_{I_1} as any of the w_I. If the vectors $w_{I_1},\dots,w_{I_j}$ with $j < \nu+1$ are given let W_j be the space generated by them. As $v_i, i = 1,\dots,n$ are a basis of V, their rest classes $\bar{v}_i \in V/W_j$ generate V/W_j. Hence there are $n-j$ natural numbers $l_j \le n$ such that $\bar{v}_{l_j}, j = 1,\dots,n-j$ form a basis of V/W_j. Since $n-j \ge n-\nu$, there is a subset I_{j+1} of $\{l_j | j = 1,\dots,n-j\}$ with $|I| = n-\nu$, such that the restclass $\bar{w}_I$ of w_I in V/W_j is nonzero. Conseqently $w_{I_1},\dots,w_{I_j}, w_{I_{j+1}}$ are linearly independent.

2. Let $X^{\bar{G}}$ be the set of characters of $\mathbb{G}_m^{mn}$ that are one on $\bar{G}$ and X^H the character module generated by the $\chi_j, j = 1,\dots,n$. If the codimension of H in $\bar{G}$ were smaller than n, the intersection of $X^{\bar{G}}$ with X^H would be nonzero. So assume that there are $r_1,\dots,r_n \in \mathbb{Z}$, not all zero, such that

$$\chi := \prod_{j=1}^n \chi_j^{r_j} = \prod_{i=1}^m \prod_{j=1}^n x_{ij}^{l_i r_j} \in X^{\bar{G}}.$$

Then, the resstriction of χ to $\bar{G}$ would be 1, hence $\chi((\exp(\theta_i\kappa_j))_{i\le m, j\le n}) = 1$. Since the θ_i, κ_j are all real, this is equivalent to

$$\sum_{i=1}^m \sum_{j=1}^n (l_i\theta_i)(r_j\kappa_j) = 0, \quad \text{hence} \quad \left(\sum_{i=1}^m l_i\theta_i\right)\left(\sum_{j=1}^n r_j\kappa_j\right) = 0.$$

which implies that either $\sum_{i=1}^m l_i\theta_i$ or $\sum_{j=1}^n r_j\kappa_j = 0$, which in turn, because of the linearily independent conditions, implies that either $l_1 = \ldots = l_m = 0$ or $r_1 = \cdots = r_n = 0$ in contradiction to the assumptions.

Proof of Theorem I Let $\Theta = (\Theta_1, \dots, \Theta_m) = \exp(\theta_1, \dots, \theta_m)$, and $x_1, \dots, x_m$ the coordinate functions of $\mathbb{G}_m^m$. For $l = (i_1, \dots, l_m) \in \mathbb{Z}^m$ denote the corresponding character $\prod_{i=1}^m x_i^{l_i}$ of $\mathbb{G}_m^m$ by χ_l; it has degree $\sum_{i=1}^m |l_i| \leq \sqrt{m}|l|$.

1. For $t = 1$, assume that $T(\Theta) = 0$, and $\text{gen}(\theta, 1) \leq m-1$. Since θ is $(m,1)$-special, for every $c > 0$ there are infinitely man $D \in \mathbb{N}$ and nonzero $l = (l_1, \dots, l_m)$ with

$$|l| \leq D, \quad \text{hence} \quad \deg \chi_l \leq \sqrt{m}|l|,$$

and

$$|\chi_l(\Theta) - 1| = |\chi_l(\Theta_1, \dots, \Theta_m) - 1| \leq \exp(-cD).$$

Since θ is regular, we also have $|\chi_l(\Theta) - 1| > 0$, and since Θ_i is algebraic for every $i = 1, \dots, m$, if c is sufficiently big, this contradicts the Liouville inequality.

For $t \geq 2$, let X be the algebraic closure of $\{(\Theta_1, \dots, \Theta_m)\}$ over $\mathbb{Q}$, and assume $T(\Theta) = \dim X \leq t-1$. Since, $\text{gen}(\theta, t) \leq m-t$, for every $c > 0$, there are infinitely many $D \in \mathbb{N}$ such that for every $1 \leq i_1 < \dots < i_{m+1-t} \leq m$, there is a nonzero $l = (l_{i_1}, \dots, l_{m+1-t}) \in \mathbb{Z}^\mu$ with $|l| \leq D$, hence $\deg \chi_l \leq \sqrt{m}D$, and

$$\log |\chi_l(\Theta_{1_1}, \dots, \Theta_{i_{m+1-t}}) - 1| \leq -cD^t.$$

By part one of the previous Lemma, the rank of the module N generated by these $(l_1, \dots, l_{m-t}) \in \mathbb{Z}^\mu$ is t. Hence, by conjecture I, there is a Zariski closed subset $\bar{X} \subset X$ such that $\bar{X}$ contains $X \cap H_N$. Thus the first part of the claim follows from Proposition 3.3. The second part of the claim is proved analogously.

2. If $\text{gen}(\theta, t) < m$, then θ is (m,t)-special. Hence, for every $c > 0$, there is an infinite subset $\mathcal{D} \subset \mathbb{N}$ such that for every $D \in \mathcal{D}$, there is an $l = l_1, \dots, l_m$ such that $\deg \chi_l \leq \sqrt{m}D$, and

$$\log |\chi_l(\exp(\theta_1), \dots, \exp(\theta_m)) - 1| \leq -cD^t.$$

Consequently,

$$\log |\chi_l(\exp(\theta_1\kappa_j), \dots, \exp(\theta_m\kappa_j)) - 1| \leq -c'cD^t,$$

with j any natural number less or equal n, and c' only depending on κ.

Next, with $\Theta = (\exp(\theta_i\kappa_j)_{i\leq m, j\leq n}$, let $X \subset \mathbb{G}_m^{mn}$ be the algebraic closure of $\{\Theta\}$, and $\bar{G} \subset \mathbb{G}_m^{mn}$ the smallest subgroup of G_m^{mn} that contains θ. Then, $\bar{G}$ is isomorphic to $\mathbb{G}_m^a$ for some $a \in \mathbb{N}$,because the $\exp(\theta_i\kappa_j)$ are all positive real numbers. Assume $\dim X = T(\Theta) < t \leq n$. Since $\chi_j = \prod_{j=1}^m x_{ij}^{l_i}, j = 1, \dots, n$, part 2 of the previous Lemma implies that the codimension of $H = \bar{G} \cap \cap_{j=1}^n \ker \chi_j$ in $\bar{G}$ is $n > \dim X$. By conjecture I, the common zeroes of N_D in X are contained in a fixed proper Zariski closed subset $\bar{X}$ of X, and the claim follows from Proposition 3.3. If $\text{gen}(\theta, t) < n$ and $m \geq t$, the claim follows analogously

3. Let $\theta_1, \ldots, \theta_n, \kappa_m, \ldots, \kappa_m$ be as in the conjecture, and $\eta = mn/(m+n)$. If θ is (m, η)-generic and κ is (n, η)-generic, then

$$\frac{\mathrm{gen}(\theta, \eta)\mathrm{gen}(\kappa, \eta)}{\mathrm{gen}(\theta, \eta) + \mathrm{gen}(\kappa, \eta)} = \frac{mn}{m+n} = \eta > \eta - \epsilon$$

for every $\epsilon > 0$, and Proposition 2.2 implies $T(\Theta) \geq \eta - 1 - \epsilon = \frac{mn}{m+n} - 1 - \epsilon$, for every $\epsilon > 0$ which in turn implies $T(\Theta) \geq \frac{mn}{m+n} - 1$.
If θ is (m, η)-special, it is also (m, t)-special with $t = [\eta]$, hence $\mathrm{gen}(\theta, t) < m$. Since $n > \frac{mn}{m+n} \geq t$, part 2 implies

$$T(\Theta) \geq t = \left[\frac{mn}{m+n}\right] \geq \frac{mn}{m+n} - 1.$$

If κ is (n, η)-special, $T(\Theta) \geq \frac{mn}{m+n} - 1$ follows in the same way.

Proof of Theorem II 1. For $t = 1$, the claim coincides with conjecture 1.3 for $t = 1$, and the proof of this claim did not use conjecture I.

For $t = 2$, let $\mathrm{gen}(\theta, 2) \leq \frac{m-2}{1} + 1 = m - 2$. Then, by Lemma 2.1.5, $\mathrm{gen}(\theta, 1) \leq \mathrm{gen}(\theta, 2) \leq m - 2 < m - 1$, and by the above $T(\Theta_1, \ldots, \Theta_m) \geq 1$. Assume that $T(\Theta_1, \ldots, \Theta_m) = 1$. As θ is $(m-1, 2)$-special, for every constant $c > 0$, there is an infinite set $\mathcal{D} \subset \mathbb{N}$, such that for every $D \in \mathcal{D}$, there are $l, \bar{l} \in \mathbb{Z}^m$, with $|l|, |\bar{l}| \leq D$, and

$$\log |\chi_l(\Theta_1, \ldots, \Theta_{m-1}) - 1| \leq -cD^2, \quad \log |\chi_{\bar{l}}(\Theta_2, \ldots, \Theta_m) - 1| \leq -cD^2.$$

Since l and $\bar{l}$ are linearily independent, [Mau], Theórme 1.2 implies that the common zeroes of χ_l and $\chi_{\bar{l}}$ are contained in a fixed finite subset $\bar{X}$ of X, and the first claim follows from proposition 3.3. The second claim is proved analogously.

Assume now $t \geq 3$, and the Theorem be true for $t - 1$. Let $\mathrm{gen}(\theta, t) \leq \frac{m-t}{t-1}$, and define $s := m - \mathrm{gen}(\theta, t) - 1$. Then,

$$m - s - 1 = \mathrm{gen}(\theta, t) \leq \frac{m-t}{t-1}.$$

This equation is equivalent to

$$m - s - 1 \leq \frac{s-t+1}{t-2} \quad \Longleftrightarrow \quad \mathrm{gen}(\theta, t) \leq \frac{s-(t-1)}{t-2}.$$

Let $\mathcal{A}$ be the set of $(s \times m)$-matrices of rank s with coefficients in $\mathbb{Q}$. By Lemma 2.1.5, for every $A \in \mathcal{A}$,

$$\mathrm{gen}(A\theta, t-1) \leq \mathrm{gen}(\theta, t) \leq \frac{s-(t-1)}{t-2}.$$

The induction hypothesis implies $T(\exp(A\theta)) \geq t-1$ for every $A \in \mathcal{A}$. Since $T(\Theta) \geq T(\Theta^A)$, we only need to derive a contradiction from the assumption that $T(\Theta) = t-1$. Assume $T(\Theta) = t-1$. Since $T(\Theta^A) = t-1$ for every $A \in \mathcal{A}$ the algebraic closure X of $\{\Theta\}$ over $\mathbb{Q}$ is s-regular. Hence, by Theorem 3.2, there is a proper Zariski closed subset $\bar{X}$ that contains all s-special subvarietes of X.
Since θ is $(m-s,t)$-special, because of $m-s > m-s-1 = \text{gen}(\theta,t)$, for every c there are infinitely many $D \in \mathbb{N}$ such that for every $I = \{i_1,\ldots,i_{n-s}\} \subset \{1,\ldots,m\}$, there is a nonzero $l_I = (l_{i_1},\ldots,l_{i_{n-s}})$ such that

$$|l_I| \leq D, \quad \text{and} \quad \log|\chi_{l_I}(\Theta_1,\cdots,\Theta_{n-s}) - 1| \leq -cD^t.$$

By Lemma 4.1, the l_I generate a submodule of rank $s+1$, and by the above he intersection of X with $\cap_I \ker(\chi_{l_I})$ is contained in $\bar{X}$. Proposition 3.3 implies $T(\exp(\theta)) \geq t$ in contradiction with the assumption.
Of course, the claim about κ is proved in the same way.

2. If $\text{gen}(\theta,t) \geq \mu$ and $\text{gen}(\theta,t) \geq \nu$, then

$$\frac{\text{gen}(\theta,t)\text{gen}(\kappa,t)}{\text{gen}(\theta,t)+\text{gen}(\kappa,t)} \geq \frac{\mu\nu}{\mu+\nu} > t,$$

hence for some $\epsilon > 0$,

$$\frac{\text{gen}(\theta,t)\text{gen}(\kappa,t)}{\text{gen}(\theta,t)+\text{gen}(\kappa,t)} > t+\epsilon,$$

and Proposition 2.2 implies $T(\Theta) \geq t+\epsilon-1$. Since both t and $T(\Theta)$ are natural numbers, this implies $T(\Theta) \geq t$.
If $\text{gen}(\theta,t) < \mu \leq \frac{m-t}{\max(1,t-1)}$, by Lemma 2.1.3, likewise $\text{gen}(\theta_1\kappa_1,\ldots,\theta_m\kappa_1,t) \leq \frac{m-t}{\max(1,t-1)}$. By part 1, $T((\exp(\theta_1\kappa_1),\exp(\theta_m\kappa_1)) \geq t$. The claim thus follows from the trivial fact $T(\Theta) \geq T((\exp(\theta_1\kappa_1),\exp(\theta_m\kappa_1))$. If $\text{gen}(\kappa,t) < \nu$, the claim is proved analogously.

A Proof of Proposition 2.2

Let $\Theta = (\exp(\theta_i\kappa_j))_{i\leq m,j\leq n} \in \mathbb{G}_m^{mn}$, and

$$\bar{\Theta}_k := (\exp(\theta_i\kappa_j))_{i\leq m,j\leq n,a\leq k} \in \mathbb{G}_m^{mnk}(\mathbb{C}),$$

where k is any natural number. Of course, $T(\bar{\Theta}_k) = T(\Theta)$, for any k.
For η as in the Proposition, let $\mu := gen(\theta,\eta), \nu := gen(\kappa,\eta)$, and for $D \in \mathbb{N}$, let

$$L = L(D) := [D^{\frac{\nu}{\mu+\nu}}], \quad R = R(D) := [(2\mu+1)D^{\frac{\mu}{\mu+\nu}}].$$

Further, for $I = (i_1,\ldots,i_\mu), J = (j_1,\ldots,m_\nu)$ with $1 \leq i_1 < \cdots < i_\mu \leq m$, and $1 \leq j_1 < \cdots < i_\nu \leq n$, let

$$p_I : \mathbb{G}_m^{mk} \to G_{I,k} \cong \mathbb{G}_m^{\mu k}, \quad (z_{ia})_{i\leq m,a\leq k} \mapsto (z_{i_\lambda a})_{\lambda\leq\mu,a\leq k},$$

$$p_J : \mathbb{G}_m^{nk} \to G_{J,k} \cong \mathbb{G}_m^{\nu k}, \quad (z_{jb})_{j\leq n, b\leq k} \mapsto (z_{j_\rho b})_{\rho\leq \nu, b\leq k},$$

which induce maps of coordinate rings

$$p_I^* : \mathbb{Z}[(\mathbb{G}_m)_{I,k}] \to \mathbb{Z}[\mathbb{G}_m^{mk}], \quad p_J^* : \mathbb{Z}[(\mathbb{G}_m)_{J,k}] \to \mathbb{Z}[\mathbb{G}_m^{nmk}],$$

For $R \in \mathbb{N}$ let $B_R := \{r \in M_{J\times k}(\mathbb{N}) | r_{j_\rho a} \leq R, \ \forall \rho = 1, \dots, \mu, a = 1, \dots, k\}$ be the matrices indexed by $J = (j_1, \dots, j_\nu)$ and $1, \dots, k$ with entries in the natural numbers less or equal to R. For $r \in B_R$, and $a \in \{1, \dots, k\}$ define

$$m_{r,a} : \mathbb{G}_m^{mnk} \to \mathbb{G}_I \cong \mathbb{G}_m^{\mu}, \quad (z_{iaj})_{i\leq m, a\leq k, j\leq n} \mapsto \left(\prod_{\rho=1}^{\nu} z_{i_\lambda a j_\rho}^{r_{j_\rho a}}\right)_{\lambda\leq\mu},$$

$$m_r : \mathbb{G}_m^{mnk} \to \mathbb{G}_{I,k} \cong \mathbb{G}_m^{\mu k}, \quad (z_{iaj})_{i\leq m, j\leq n, a\leq k} \mapsto \left(\prod_{\rho=1}^{\nu} z_{i_\lambda a j_\rho}^{r_{j_\rho a}}\right)_{\lambda\leq\mu, a\leq k},$$

with corresponding coordinate maps

$$m_{r,q}^* : \mathbb{Z}[\mathbb{G}_I] \to \mathbb{Z}[\mathbb{G}_m^{mnk}], \quad m_r^* : \mathbb{Z}[\mathbb{G}_{I,k}] \to \mathbb{Z}[\mathbb{G}_m^{mnk}].$$

We have

$$\deg m_{r,a}^*(f) \leq \max_{\rho=1,\dots,\nu} |r_{j_\rho a}| \deg f, \quad \deg m_r^*(g) \leq \max_{\rho=1,\dots,\nu, a=1,\dots,k} |r_{j_\rho a}| \deg g, \tag{1}$$

and

$$\log |m_{r,a}^*(f)| \leq \log |f|, \quad \log |m_r^*(g)| \leq \log |g|, \tag{2}$$

for all $f \in \mathbb{Z}[\mathbb{G}_I], g \in \mathbb{Z}[\mathbb{G}_{I,k}]$.
Finally, for $\mathbf{z} = (z_{iaj})_{iaj} \in \mathbb{G}_m^{mnk}$, $a \leq k$, and $R \in \mathbb{N}$, define

$$\Sigma_{R,a}(\mathbf{z}) := \{m_{r,a}(\mathbf{z}) | \ r \in B_R\} \subset G_I(\mathbb{C}), \quad a = 1, \dots, k,$$

$$\Sigma_R(\mathbf{z}) := \{m_r(\mathbf{z}) | \ r \in B_R\} = \Sigma_{R,1}(\mathbf{z}) \times \cdots \times \Sigma_{R,k}(\mathbf{z}) \subset G_{I,k}(\mathbb{C}).$$

A.1 Propostion *For $G = \mathbb{G}_m^m$, and $\bar{\Sigma} \subset G(\mathbb{C})$ a finite subset, define*

$$\Sigma = \Sigma(d) = \{\sigma_1 \cdot \dots \cdot \sigma_d | \sigma_i \in \bar{\sigma}, i = 1, \dots, d\},$$

and assume there is a polynomial $f \in \mathbb{C}[z_1, \dots, z_m]$ of degree at most L that vanishes on every point of Σ. Then there is a proper subgroup $H \subset G$ such that

$$card(\Sigma H/H)\mathcal{H}_H(L) \leq \mathcal{H}_G(L),$$

where $\mathcal{H}_H, \mathcal{H}_G$ are the Hilbert functions of H and G. Moreover H can be chosen as the subgroup given by an equation $\prod_{i=1}^m x_i^{l_i} = 1$ where $|l_i| \leq L$ for every $i = 1, \dots, m$.

PROOF In [LNM 1752], ch. 11, Theorem 4.1 take $T = 0$.

A.2 Lemma *For $l \in \mathbb{N}$, and Σ a finite subset of $\mathbb{C}^l$ define*

$$\omega(\Sigma) := \min\{\deg P | P \in \mathbb{C}[z_1, \dots, z_l], P \neq 0, P(\sigma) = 0\ \forall \sigma \in \Sigma.\}$$

Then, for $\Sigma_1, \dots, \Sigma_k$ finite subsets of $\mathbb{C}^\mu$,

$$\omega(\Sigma_1 \times \dots \times \Sigma_k) = \min_{1 \leq a \leq k} \omega(\Sigma_a).$$

PROOF [LNM 1752], ch. 14, Proposition 3.3.

A.3 Lemma *For $\eta > 1$ assume (θ, κ) is (ν, μ, η)-regular, and $\mathbf{z} = (z_{iaj})_{iaj} = (\mathbf{z}_a)_{a \leq k} \in \mathbb{G}^{mnk}(\mathbb{C})$ is any point. For every sufficiently big D, there are subsets $I = \{i_1, \dots, i_\nu\} \subset \{1, \dots, n\}$ and $J = \{j_1, \dots, i_\mu\} \subset \{1, \dots, m\}$ such that the existence of a polynomial $f \in \mathbb{Z}[(\mathbb{G})_I]$ with $\deg f \leq L(D)$ that is zero on every point of $\Sigma_{\left[\frac{R(D)}{2}\right]}(\mathbf{z})$, implies $\log |\Theta_k, \mathbf{z}| \gg -D^\eta$, where $|\bar{\Theta}_k, \mathbf{z}|$ is the distance of $\bar{\Theta}_k$ to $\mathbf{z}$, and the implied constant depends only on m, n, k and Θ.*

PROOF Let $(l_\lambda)_\lambda \in \mathbb{Z}^\mu \setminus \{0\}, (r_\rho)_\rho \in \mathbb{Z}^\nu \setminus \{0\}$, $I \subset \{1, \dots, m\}$, $J \subset \{1, \dots, n\}$, and $a \leq k$. Then,

$$\prod_{\lambda=1}^{\mu} \prod_{\rho=1}^{\nu} (\bar{\Theta}_k)_{i_\lambda a j_\rho}^{l_\lambda r_\rho} = \prod_{\lambda=1}^{\mu} \prod_{\rho=1}^{\nu} \Theta_{i_\lambda j_\rho}^{l_\lambda r_\rho} = \exp\left(\left(\sum_{\lambda=1}^{\mu} l_\lambda \theta_{i_\lambda}\right)\left(\sum_{\rho=1}^{\nu} r_\rho \kappa_{j_\rho}\right)\right).$$

If $|l_\lambda| \leq L, \forall \lambda$, and $|r_\rho| \leq R, \forall \rho$, since $L = L(D), R = R(D) \to \infty$ when $D \to \infty$, the (ν, μ, η)-regularity of Θ implies that for any sufficiently big D there are I, J such that for all $(l_\lambda)_\lambda \in \mathbb{Z}^\mu \setminus \{0\}, (r_\rho)_\rho \in \mathbb{Z}^\nu \setminus \{0\}$ with $|l_\lambda| \leq L, \forall \lambda$ and $|r_\rho| \leq R, \forall \rho$, the inequality

$$\log \left| \prod_{\lambda=1}^{\mu} \prod_{\rho=1}^{\nu} (\bar{\Theta}_k)_{i_\lambda a j_\rho}^{l_\lambda r_\rho} - 1 \right| \geq \log \left| \sum_{\lambda=1}^{\mu} l_\lambda \theta_{i_\lambda} \right| \left| \sum_{\rho=1}^{\nu} r_\rho \kappa_{j_\rho} \right| + \log 2 \gg$$

$$L^\eta - R^\eta \geq -(2k(m+n)D)^\eta \gg -D^\eta \tag{3}$$

holds, again because of $\exp(0) = \exp'(0) = 1$. (Strictly speaking the inequality holds only if the argument of the exponential is sufficiently close to 0, but these are the only arguments we are concerned with.)
Assume now that there is an $\bar{f} \in \mathbb{Z}[(\mathbb{G})_{I,k}]$ with $\deg \bar{f} \leq L$ that is zero at every point of $\Sigma_{\left[\frac{R}{2\mu}\right]}(\mathbf{z})$. By Lemma A.2, there is an $a \leq k$ and an $f \in \mathbb{Z}[(\mathbb{G})_I]$ with $\deg f \leq L$

that is zero at every point at $\Sigma_{[\frac{R}{2\mu}],a}(\mathbf{z})$. By Propostion A.1 there is a subgroup H of G_I such that

$$\mathrm{Card}((\Sigma_{[\frac{R}{2\mu}],a}(\mathbf{z})H)/H)\mathcal{H}_H(L) \leq \mathcal{H}_{G_I}(L),$$

Morover, H can be chosen to be defined by an equation

$$\prod_{\lambda=1}^{\mu} x_{i_\lambda a}^{l_\lambda} = 1 \quad \text{with} \quad |l_\lambda| \leq L,\ \lambda = 1, \ldots, \mu, \exists \lambda : l_\lambda \neq 0.$$

As $G_I \cong \mathbb{G}_m^\mu$, we have $\mathcal{H}_{G_I}(L) = L^\mu$, hence $\mathrm{Card}((\Sigma_{[\frac{R}{2\mu}],a}(\mathbf{z})H)/H) \leq L^\mu$. As because of $\frac{\mu}{\mu+\nu} > \eta > 1$, for sufficiently big D,

$$\mathrm{Card}(\Sigma_{[\frac{R}{2\mu}],a}(\mathbf{z})) = \left[\frac{R}{2\mu}\right]^\nu > [D^{\frac{\mu}{\mu+\nu}}]^\nu = L^\mu,$$

this implies that there are two different points $\sigma, \bar{\sigma} \in \Sigma_{[\frac{R}{2\mu}],a}(\mathbf{z})$ such that $\sigma\bar{\sigma}^{-1} \in H$. With $\sigma = m_{r,a}(\mathbf{z}), \bar{\sigma} = m_{\bar{r},a}(\mathbf{z})$, we have $\sigma\bar{\sigma}^{-1} = m_{r,a}(\mathbf{z}) \cdot m_{-\bar{r},a}(\mathbf{z})$, hence

$$\prod_{\lambda=1}^{\mu}\prod_{\rho=1}^{\nu} z_{i_\lambda a j_\rho}^{l_\lambda(r_{j_\rho a} - \bar{r}_{j_\rho a})} = 1. \tag{4}$$

Since $\sigma \neq \bar{\sigma}$, there is an a and a ρ such that $r_{j_\rho a} \neq \bar{r}_{j_\rho a}$, and since $\sigma, \bar{\sigma} \in \Sigma_{[\frac{R}{2\mu}],a}(\mathbf{z})$, the inequalty $|r_{j_\rho a} - \bar{r}_{j_\rho a}| \leq \frac{R}{\mu}$ holds, for every $\rho = 1, \ldots, \nu$.
Assume $\log|\bar{\Theta}_k, \mathbf{z}| \leq -cD^\eta$ with some arbitrarily big constant c. Since the imaginary part of $(\bar{\Theta}_k)_{iaj}$ is zero for every i, a, j, this implies that there are $\mathbf{x}_{iaj} \in \mathbb{C}, i = 1, \ldots, m, a = 1, \ldots, k, j = 1, \ldots, n$ such that $\mathbf{z}_{iaj} = \exp(x_{iaj})$ and

$$\log|\mathrm{im}\ x_{iaj}| \leq -cD^\eta + \log 2, \quad \forall i = 1, \ldots, m,\ a = 1, \ldots, k,\ j = 1, \ldots, n,$$

consequently because of $\eta > 1$, for D sufficiently big

$$K \log|\mathrm{im}\ x_{iaj}| \leq -c'D^\eta + \log 2, \quad \forall i = 1, \ldots, m,\ a = 1, \ldots, k,\ j = 1, \ldots, n,$$

for every number $K \leq 2\mu\nu LR$. Thus, because of (4),

$$\log\left|\prod_{\lambda=1}^{\mu}\prod_{\rho=1}^{\nu}(\bar{\Theta}_k)_{i_\lambda a j_\rho}^{l_\lambda(r_{j_\rho a} - \bar{r}_{j_\rho a})} - 1\right| =$$

$$\log\left|\prod_{\lambda=1}^{\mu}\prod_{\rho=1}^{\nu}(\bar{\Theta}_k)_{i_\lambda a j_\rho}^{l_\lambda(r_{j_\rho a} - \bar{r}_{j_\rho a})} - \prod_{\lambda=1}^{\mu}\prod_{\rho=1}^{\nu}\exp(x_{i_\lambda a j_\rho})^{l_\lambda(r_{j_\rho a} - \bar{r}_{j_\rho a})}\right| \leq$$

$$\log\left|\sum_{\lambda=1}^{\mu}\sum_{\rho=1}^{\nu} l_\lambda(r_{j_\rho a} - \bar{r}_{j_\rho a})(\theta_{i_\lambda}\kappa_{j_\rho} - \mathrm{Re}\ \mathbf{x}_{i_\lambda a j_\rho}) + \sum_{\lambda=1}^{\mu}\sum_{\rho=1}^{\nu} l_\lambda(r_{j_\rho a} - \bar{r}_{j_\rho a})\mathrm{im}\ x_{iaj}\right| \leq$$

$$\max(\log(2\mu\nu LR) + \log|\bar{\Theta}_k, \mathbf{z}|, -c'D^\eta + \log 2) + \log 2,$$

which by assumption is less or equal, which for sufficiently big D is less or equal

$$-c''D^\eta,$$

where c'' is a constant depending only on m, n, k, andΘ times the arbitrarily big chosen constant c, which because of $\eta > 1$ for $D \gg 0$ contradicts (3).

To proof the proposition, a series of auxiliary polynomials fulfilling the conditions in Proposition 1.7 will be construed. The main tool for this is following Lemma.

A.4 Lemma *For $r < 0$ and an holomorphic function $\varphi : \mathbb{C}^k \to \mathbb{C}$ let*

$$|\varphi|_r := \sup_{|z_i| \leq r, i=1,\dots k} \varphi(z).$$

Let further $M \in \mathbb{N}$, and Δ, U be positive real numbers. If $(8U)^{k+1} \leq M\Delta$ and $\Delta \leq U$, then for any holomorphic functions $\varphi_1, \dots, \varphi_M : \mathbb{C}^k \to \mathbb{C}$ with

$$\sum_{l=1}^{M} |\varphi_l|_{er} \leq \exp(U),$$

there are numbers $h_1, \dots, h_M \in \mathbb{Z}$ with $\log|h_l| \leq \Delta, l = 1, \dots M$ such that the function $\varphi = \sum_{l=1}^{M} h_l \varphi_M$ satisfies $\log|\varphi|_r \leq -U$.

Proof [Wal]

For $X = (x_{i_\lambda a})_{\lambda \leq \mu, j \leq k}$, and A_L the set of multidegrees $d = (d_{\lambda a})_{\lambda \leq \mu, a \leq k}$ denote $|d| := \sum_{\lambda=1}^{\mu} \sum_{a=1}^{k} d_{\lambda a}$, and let

$$A_L := \left\{ X^d = \prod_{\lambda \leq \mu, a \leq k} x_{i_\lambda a}^{d_{\lambda a}} \middle| \, |d| \leq L \right\}$$

be the monomials of degree at most L in $\mathbb{Z}[\mathbb{G}^{mk}]$, that lie in the image of $p_I^* : \mathbb{Z}[G_{I,k}] \to \mathbb{Z}[\mathbb{G}_m^{mk}]$. Further,

$$i_\theta : \mathbb{C}^k \to \mathbb{G}_m^{mk}(\mathbb{C}), \quad (z_1, \dots, z_k) \mapsto (\exp(\theta_i z_j))_{i \leq m, j \leq k},$$

and for $X^d \in A_L$,

$$\varphi_d : \mathbb{C}^k \to \mathbb{C}, \quad \mathbf{z} \mapsto (X^d \circ i_\theta)(\mathbf{z}).$$

Further, put

$$M := |A_L| = \binom{L + \mu k - 1}{\mu k} \geq \frac{L^{\mu k}}{(\mu k)!}, \quad \Delta := D \leq LR, \quad \text{and} \quad U = \frac{(M\Delta)^{\frac{1}{k+1}}}{8}.$$

As $(8U)^{k+1} = M\Delta,\ \Delta \leq U$, and for all $d \in A_L$, the inequalities

$$\sup_{|\mathbf{z}_a| \leq eR\nu k|\kappa|,\ a=1,\ldots,k} \log\left(\sum_{d \in A_L} |\varphi_d((z_a)_{a\leq k})|\right) + \log M \leq$$

$$\sup_{|\mathbf{z}_a| \leq eR\nu k|\kappa|,\ a=1,\ldots,k} \sum_{\lambda \leq \mu, a \leq k} d_{\lambda a} \log|\theta_{i_\lambda} z_a| + \log|A_L| + \log M \leq$$

$$eLRmnk|\theta||\kappa| + 2\log M \leq (mnk(2\mu+1)|\theta||\kappa| + 1)D \leq U$$

hold. The last two inequalities are valid for sufficiently big D because of $LR \leq (2\mu+1)D$, and $\log M \ll L \leq D$ for big D. The Lemma above thus implies that for every $X^d \in A_L$ there is an $h_d \in \mathbb{Z}$ with $\log|h_d| \leq \Delta$ such that $\varphi = \sum_{d \in A_L} h_d \varphi_d = \sum_{d \in A_L} h_d (X^d \circ i_\theta)$ fulfills

$$\sup_{|(z_a)_{a\leq k}| \leq R\nu k|\kappa|} \log|\varphi| \leq -U = -\frac{(M\Delta)^{\frac{1}{k+1}}}{8} \leq -\frac{1}{8((\mu k)!)^{\frac{1}{k+1}}} D^{\frac{\mu\nu}{\mu+\nu}\frac{k}{k+1}}, \tag{5}$$

where $L = [D^{\frac{\nu}{\mu+\nu}}]$ was used. Define $f = \sum_{d \in A_L} X^d$ so that $\varphi = f \circ i_\theta$.
Since for $r \in B_R$, with r^t the transpose of r, the inequality

$$|r^t\kappa| = \left|\left(\sum_{\rho=1}^{\nu} r_{j_\rho a}\kappa_{j_\rho}\right)_{a\leq k}\right| \leq k\nu R|\kappa|$$

holds, by (5)

$$\log|(m_r^* f)(\bar\Theta_k)| = \log|f(m_r(\bar\Theta_k))| = \log|f((\bar\Theta_k)_{i_\lambda a j_\rho}^{r_{j_\rho a}})_{\lambda\leq\mu, a\leq k})| =$$

$$\log|f((\theta_{i_\lambda}\kappa_{j_\rho})^{r_{j_\rho a}})_{\lambda\leq\mu, a\leq k})| = \log\left|\varphi\left(\left(\sum_{\rho=1}^{\nu} r_{i_\rho a}\kappa_{j_\rho}\right)_{a\leq k}\right)\right| =$$

$$\log|\varphi(r^t\kappa)| \leq -c_2\left(D^{\frac{\mu\nu}{\mu+\nu}}\right)^{\frac{k}{k+1}},$$

for all $r \in B_R$.
By the assumption on η

$$\frac{\mu\nu}{\mu+\nu} = \frac{\mathrm{gen}(\theta,\eta)\mathrm{gen}(\kappa,\eta)}{\mathrm{gen}(\theta,\eta) + \mathrm{gen}(\kappa,\eta)} > \eta.$$

Thus, for a sufficiently big k, also $\left(\frac{\mu\nu}{\mu+\nu}\right)^{\frac{k}{k+1}} > \eta$
Hence for an arbitrarily small $\epsilon > 0$ and a sufficiently big D,

$$\log|(m_r^* f)(\bar\Theta_k)| \leq -c_2\left(D^{\frac{\mu\nu}{\mu+\nu}}\right)^{\frac{k}{k+1}} \leq -c_2 D^\eta \leq -cD^{\eta-\epsilon}. \tag{6}$$

for all $R \in B_R$, and every $c > 0$.
Assume that there is a point $z \in \mathbb{G}_m^{mnk}(\mathbb{C})$ such that

$$f(m_r(z)) = ((m_r^* f)(z) = 0,$$

for all $r \in B_R$. Then, by Lemma 2.1.6 Θ is (μ, ν, η)-regular, and Lemma A.3 implies

$$\log |\bar{\Theta}_k, z| \gg -3D^\eta,$$

hence for a sufficiently big D,

$$\log |\bar{\Theta}_k, z| \geq -cD^{\eta-\epsilon}.$$

Thus, the distance of $\bar{\Theta}_k$ to any common zero of the set $\{m_r f | r \in B_R\}$ is at least $-cD^{\eta-\epsilon}$. Because of (6), to apply Proposition 1.7, it only remains to check the upper bounds on the degree and length of the $m_r^* f$. By construction $\deg f \leq L, \log |f| \leq LR$. Hence, by (1),

$$\deg m_r^* f \leq L \max_{\rho_1,\dots,\nu,a=1,\dots,a} r_{j_\rho a} \leq LR \leq (2\mu + 1)D,$$

and by (2)

$$\log |m_r^* f| \leq LR \leq c'(2\mu + 1)D.$$

As all conditions of Proposition 1.7 are fulfilled, $T(\bar{\Theta}_k) \geq \eta - \epsilon - 1$ follows. Since $T(\bar{\Theta}_k)$ is integral, for a sufficiently small ϵ, this implies $T(\bar{\Theta}_k) \geq \eta - 1$, hence

$$T(\Theta) = T(\bar{\Theta}_k) \geq \eta - 1.$$

References

[Ha] P. Habegger: On the bounded height conjecture. International Mathematics Research Notices, Volume 2009, Issue 5, 2009, Pages 860–886,

[LNM 1752] Introduction to algebraic independence theory. Springer Lecture Notes 1752

[Ma] H. Massold: Intersections of subvarietes of $\mathbb{G}_m^n$ with algebraic subgroups. To appear

[Mau] G. Maurin: Courbes algébrique et équations multiplicatives. Mathematische Annalen, Volume 341, pages 789–824, (2008)

[Ph] P. Phillippon: Critères pour l'indépendance algébriques, Inst. hautes études sci. publ. mathématiques 64 (1986), 5-52

[Wal] M. Waldschmidt: Transcendance et exponentielles en plusieurs variables, Invent. Math. 63/1, (1981), 97-127